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More on the Analysis of Local Regularity Through Wavelets

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In this paper we want to extend the results of pointwise analysis through wavelet transforms to the class of functions where the local fluctuation is bounded by any submultiplicative function. This generalizes the results obtained before in the well-known case of Hölder regularity.

KEY WORDS: Fractals; wavelets; local regularity.

1. INTRODUCTION

We may distinguish at least three aspects in fractals: (i) an underlying dynamical system, (ii) global self-similarity, and (iii) local self-similarity. Whereas for the first two points, we do not know yet how to use in general the wavelet technique as analysis tool, the third point is quite settled now. In particular we have strong theorems about the analysis of local regularity through wavelet transforms. The oldest theorems in that directed used the order of magnitude of the gradient of the harmonic extension of a function over the real line to characterize global Hölder regularity (e.g., ref. 1). Recently, using the wavelet transform, sufficient conditions on the wavelet side have been found to prove pointwise Hölder regularity of functions and even pointwise differentiability.[2,3] All these theorems are of Tauberian nature, that is, they are optimal, but they do not give a complete characterization of pointwise regularity.

In this paper we propose to extend the theorem of pointwise Hölder regularity to a larger class of pointwise regularity, where the local fluctuations are bounded by some submultiplicative function. We will then apply

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the wavelet technique to the Brownian motion, where we shall redemonstrate Levi's law and to the Weierstrass function for which we can easily prove its nondifferentiability. In a second part we shall introduce a more general concept of local self-similarity linked to the notion of local renormalization in wavelet space.

2. WAVELET TRANSFORMS

The wavelet transform has by now become a well-known tool in analysis (see, e.g., refs. 4–6 and 15). In the case of its application to fractals it has been shown that it works as a mathematical microscope[7-9] that reveals the small-scale features of the analyzed function. We now start by recalling the properties of the wavelet transform that are necessary for this paper. For the presentation of the wavelet transform in an L^2 context we refer to ref. 10.

The wavelet transform of $s \in L^1(\mathbb{R})$ with respect to the analyzing wavelet $g \in L^1(\mathbb{R})$ is defined as the following set of convolutions:

$$\mathcal{W}_g s(b, a) = \int_{-\infty}^{+\infty} dt \frac{1}{a} \bar{g}\left(\frac{t-b}{a}\right) s(t) = \langle g_{b,a} | s \rangle = \tilde{g}_a * s(b)$$

where we have introduced some notations that we shall systematically use in the sequel:

$$\tilde{g}(t) = \bar{g}(-t), \qquad g^{\dagger}(t) = g(-t), \qquad g_a = g(\cdot/a)/a, \qquad g_{b,a} = g_a(\cdot - b)$$

The wavelet transform thus maps functions over the real line $\mathbb{R}$ to functions over the half-plane H, which may be interpreted as a position-scale half-plane, and the wavelet transform thus is a sort of mathematical microscope.

Although a single convolution may happen not to be invertible, the set of convolutions defining a wavelet transform always has an inverse: let $h \in L^1(\mathbb{R})$ satisfy

$$\log(2 + |t|) h * \tilde{g} \in L^1 \tag{2.1a}$$

$$\int_0^{\infty} \frac{d\omega}{\omega} \hat{h}(\omega) \hat{\bar{g}}(\omega) = \int_0^{\infty} \frac{d\omega}{\omega} \hat{h}(-\omega) \hat{\bar{g}}(-\omega) = c_{g,h}, \qquad 0 < |c_{g,h}| < \infty \tag{2.16}$$

where the Fourier transform is defined as

$$\hat{g}(\omega) = \int_{-\infty}^{+\infty} dt \, e^{-i\omega t} g(t)$$

Such a function h is called a reconstruction wavelet for g. Suppose further that $s \in L^{\infty}(\mathbb{R})$ is weakly oscillating around 0 in the following sense:

$$\frac{1}{2\alpha}\int_{t-\alpha}^{t+\alpha} s \to 0, \qquad \alpha \to \infty \qquad \text{uniformly in } t \tag{2.2}$$

Then we have[2]

$$\begin{aligned} s_{\varepsilon,\rho}(t) &= \frac{1}{c_{g,h}}\int_{\varepsilon}^{\rho}\frac{da}{a}\int_{-\infty}^{+\infty} db\, \mathscr{W}_g s(b,a)\frac{1}{a}h\left(\frac{t-b}{a}\right) \\ &= \frac{1}{c_{g,h}}\int_{\varepsilon}^{\rho}\frac{da}{a}\,\tilde{g}_a * h_a * s(t) \to s(t), \qquad \varepsilon \to 0, \quad \rho \to \infty \end{aligned} \tag{2.3}$$

It can be shown[3] that under the hypothesis on g and h there is a function $r \in L^1(\mathbb{R})$ such that

$$s_{\varepsilon,\rho} = r * s(\cdot/\varepsilon)/\varepsilon - r * s(\cdot/\rho)/\rho$$

Thus for weakly oscillating $s \in L^{\infty}$ the convergence holds pointwise in every point of continuity and $\|s_{\varepsilon,\rho}\|_{\infty}$ is uniformly bounded. If in addition s is uniformly continuous, then the convergence holds uniformly in L^{∞}. Note that if s has only positive frequencies, then clearly only the positive-frequency part of (2.1) should hold.

Other functional contexts are possible. Consider the functions in the class of Schwarz $S(\mathbb{R})$ consisting of those functions that together with their derivatives decay faster than any polynomial. Let $S_+(\mathbb{R})$ be the subset of $S(\mathbb{R})$ consisting of those functions whose Fourier transform is supported by the positive frequencies only. For s, g, $h \in S_+(\mathbb{R})$ the inversion formula (2.3) converges in the topology of $S(\mathbb{R})$ (e.g., ref. 11). Actually it is not necessary that h and g are both in $S_+(\mathbb{R})$. Essentially all that is really needed is $\tilde{g} * h \in S_+(\mathbb{R})$. It can easily be shown that the wavelet transform of $s \in S_+(\mathbb{R})$ with respect to $g \in S_+(\mathbb{R})$ is very well localized over the half-plane. More precisely, it satisfies

$$|\mathscr{W}_g s(b,a)| \leqslant O\left(\frac{1}{(1+|b|)^m\,(a+a^{-1})^m}\right) \tag{2.4}$$

for all $m > 0$. On the other side of duality let $\eta \in S'(\mathbb{R})$ be a tempered distribution. Let $\mathscr{W}_g \eta(b,a) = \eta(\bar{g}_{b,a})$ be the wavelet transform of η with respect to $g \in S_+(\mathbb{R})$. This is a C^{∞} function over the half-plane bounded by some polynomial in $a + 1/a$ and b, and the action of η on $s \in S_+(\mathbb{R})$ may

[2] Note that for every a the integral over b is again a well-defined convolution.

be written as an absolutely convergent integral over the half-plane (e.g., ref. 11)

$$\eta(s) = c_{g,h} \int_0^{\infty} \frac{da}{a} \int_{-\infty}^{+\infty} db\, \mathcal{W}_h^* s(b, a)\, \mathcal{W}_g^* \eta(b, a) \tag{2.5}$$

Clearly the analog is true for S_-, the space of functions in S having only negative frequencies, as well as for $S_0 = S_+ \cup S_-$.

Functions, Measures, Distributions. Note that whenever one works with wavelets $g \in S_+(\mathbb{R})$ the frontier between measures, functions, and distributions is very little natural. Indeed the derivation operator ∂_t: $S'_+(\mathbb{R}) \mapsto S'_+(\mathbb{R})$ has a unique inverse in $S'_+(\mathbb{R})$—a space defined modulo the polynomials—and we have

$$\mathcal{W}[g, s](b, a) = a^n [(i\partial_t)^n g, (i\partial_t)^{-n} s](b, a) \tag{2.6}$$

for all $n \in \mathbb{Z}$. But now for every distribution with compact support there is some primitive that actually is a function (e.g., ref. 12). Therefore we may limit ourselves to functions without any loss of generality.

3. THE ORDER OF MAGNITUDE OF WAVELET COEFFICIENTS

As we have seen in the previous section, the rapid decrease of the Fourier transform of the analyzed function at infinity and the rapid decrease of the wavelet at the zero frequency is mirrored in a rapid decrease of the wavelet coefficients at small scale. In terms of the time representation this means that a high global regularity of the analyzed functions and many vanishing moments of the wavelet imply a rapid decrease of the wavelet coefficients. This relation shall be clarified in this section. In particular, we shall show that a local regularity of s implies a local decrease of the wavelet coefficients.

In order to measure the regularity of a function s in a point t_0 we estimate how well it can be approximated locally by a polynomial P_n of degree n. That is, we suppose that we may write

$$s(t_0 + t) = P_n(t) + s_{\text{loc}}(t) \tag{3.1}$$

where the remainder becomes small compared to the principal term P_n; that is $s_{\text{loc}}(t) = o(t^n)$ as $t \to 0$.

Example 3.1. If s is continuous in t_0 we may write $s(t) = s(t_0) + o(1)$ and thus $P_0(t) = s(t_0)$. If s is differentiable at t_0, then

$s(t_0+t)=s(t_0)+ts'(t_0)+o(t)$, and therefore we may choose $P_1(t)=s(t_0)+ts'(t_0)$.

In general the faster $s_{\text{loc}}(t)$ goes to 0 together with t, the higher is the regularity of s at t_0. Therefore we introduce a hierarchy of functions with which we can compare the remainder s_{loc}.

Definition 3.2. Let s be a function that satisfies (3.1) with some $n \in \mathbb{N}_0$. We say s is of regularity Λ^α at t_0, $n<\alpha \leqslant n+1$, iff

$$s_{\text{loc}}(t)=O(t^\alpha) \qquad (t \to 0)$$

We say s is of regularity λ^α, $n \leqslant \alpha < n+1$, if

$$s_{\text{loc}}(t)=o(t^\alpha) \qquad (t \to 0)$$

We say s is of regularity $\Lambda^{\alpha,\beta}_{\log}$, $n<\alpha\leqslant n+1$, $\beta>0$, if

$$s_{\text{loc}}(t)=O(t^\alpha \log^\beta t) \qquad (t \to 0)$$

We say is of regularity $\lambda^{\alpha,\beta}_{\log}$, $n<\alpha\leqslant n+1$, $\beta>0$, if

$$s_{\text{loc}}(t)=o(t^\alpha \log^\beta t) \qquad (t \to 0)$$

We say s is in $\Lambda^\alpha(\mathbb{R})$, $\lambda^\alpha(\mathbb{R})$, $\Lambda^{\alpha,\beta}_{\log}(\mathbb{R})$, $\lambda^{\alpha,\beta}_{\log}(\mathbb{R})$ if $|s(t)| \leqslant c(1+|t|^\alpha)$ and the respective local estimations hold uniformly in t_0.

We have the following scale of local regularities ($\alpha<\alpha'$, $\beta<\beta'$, $\gamma>0$)

$$\Lambda^{\alpha,\beta}_{\log} \Leftarrow \Lambda^{\alpha,\beta'}_{\log} \Leftarrow \Lambda^{\alpha} \Leftarrow \lambda^{\alpha,\beta}_{\log} \Leftarrow \lambda^{\alpha,\beta'}_{\log} \Leftarrow \lambda^{\alpha} \Leftarrow \Lambda^{\alpha',\gamma}_{\log}$$

The local regularity of the analyzed function implies a decrease of the wavelet transform at small scales whenever the analyzing wavelet is localized and has a sufficient number of moments vanishing. This decrease is of the same type as the decrease of s_{loc}, the difference of s and its local polynomial approximation P_n.

Theorem 3.3. Let s be a polynomial bounded function

$$|s(t)| \leqslant c(1+|t|^\alpha)$$

If s is of regularity Λ^α at some τ_0, then the wavelet transform $\mathcal{W}_g s$ of s with respect to the wavelet g satisfies

$$\mathcal{W}_g s(\tau_0+b, a)=O(a^\alpha+|b|^\alpha) \qquad (b, a \to 0)$$

If s is of regularity λ^{α}, then

$$\mathscr{W}_g s(\tau_0 + b, a) = o(a^{\alpha} + |b|^{\alpha}) \qquad (b, a \to 0)$$

In both cases the wavelet should satisfy $g \in L^1(\mathbb{R})$ and $t^{\alpha} g \in L^1(\mathbb{R})$. If s is of regularity $\Lambda_{\log}^{\alpha,\beta}$, then

$$\mathscr{W}_g s(\tau_0 + b, a) = O(a^{\alpha} \log^{\beta} a + |b|^{\alpha} \log^{\beta} |b|) \qquad (b, a \to 0)$$

If s is of regularity $\lambda_{\log}^{\alpha,\beta}$, then

$$\mathscr{W}_g s(\tau_0 + b, a) = o(a^{\alpha} \log^{\beta} a + |b|^{\alpha} \log^{\beta} |b|) \qquad (b, a \to 0)$$

In both cases, the wavelet should be localized such that $g \in L^1(\mathbb{R})$ and $t^{\alpha} \log^{\beta} t g \in L^1(\mathbb{R})$. In addition, in all four cases the first n moments of g should vanish,

$$\int_{-\infty}^{+\infty} dt\, t^m g(t) = 0 \qquad \text{for} \quad m = 0, 1, \ldots, n$$

where n is the only integer that satisfies $\alpha \leqslant n < \alpha + 1$.

This theorem shows that, e.g., the local Hölder regularity of degree α is mirrored by a decrease of order a^{α} along every straight line in the half-plane passing through the point τ_0; that is, for fixed $(b, a) \in \mathbb{H}$ we have

$$\mathscr{W}_g s(\tau_0 + \lambda a, \lambda b) = O(\lambda^{\alpha}) \qquad (\lambda \to 0)$$

We will prove the theorem for an even larger class of local regularities.

Definition 3.4. A nonnegative, nondecreasing function r over $\mathbb{R}^+$ is called submultiplicative if there is a constant $c > 0$ such that

$$r(tu) \leqslant c r(t) r(u)$$

for all $t, u \in \mathbb{R}^+$. A function s over $\mathbb{R}$ is called submultiplicative if $s(t)$ and $s(-t)$, $t \geqslant 0$, are submultiplicative.

Note that the submultiplicative precisely means that $t \mapsto \log(r(e^t))$ is subadditive.

We remark that for all regularity classes we have encountered so far, the local function s_{loc} may be majorized by a submultiplicative function. Since the wavelet is operating module polynomials, it only sees the local fluctuation s_{loc}, but not the polynomial part.

Theorem 3.5. Let r be a submultiplicative function and let s satisfy

$$s(\tau_0 + t) = P_n(t) + O(r(|t|)), \qquad (t \in \mathbb{R})$$

with some polynomial P_n of degree n. Suppose the analyzing wavelet satisfies

(i) $g \in L^1(\mathbb{R})$, $r \cdot g \in L^1(\mathbb{R})$.

(ii) $\int_{-\infty}^{+\infty} dt\, g(t)\, t^m = 0,\ m \in \{0, 1,..., n\}$.

Then

$$\mathscr{W}_g s(\tau_0 + b, a) = O(r(b) + r(a))$$

uniformly in $\mathbb{H}$. If s satisfies in addition

$$s(\tau_0 + t) = P_n(t) + o(r(|t|)) \qquad (t \to 0)$$

then we have in addition

$$\mathscr{W}_g s(\tau_0 + b, a) = O(r(b) + r(a))$$

Proof. The argument is well known in the Λ^α case and may be merely translated to the general submultiplicative situation. Since under the conditions stated in the theorem the wavelet does not see the polynomial part, the theorem follows from the next two lemmas.

Lemma 3.6. Let $s(t) = s(-t)$ be an even, submultiplicative function. Then the wavelet transform of s satisfies uniformly in $(b, a) \in \mathbb{H}$

$$\mathscr{W}_g s(b, a) = O(s(b) + s(a))$$

The analyzing wavelet should satisfy $g \in L^1(\mathbb{R})$ and $sg \in L^1(\mathbb{R})$.

Proof. First note that by symmetry and monotony we have

$$s(t + u) = s(|t + u|) \leqslant s(|t| + |u|)$$

Now either $2\,|t| > |t| + |u|$ or $2\,|u| > |t| + |u|$. Therefore by monotony and submultiplicativity

$$s(t + u) \leqslant s(2\,|t|) + s(2\,|u|) \leqslant c(s(|t|) + s(|u|))$$

Therefore we may estimate

$$|\mathcal{W}_g s(b,a)| = \left| \int_{-\infty}^{+\infty} dt\, \frac{1}{a}\, \bar{g}\left(\frac{t-b}{a}\right) s(t) \right|$$

$$= \left| \int_{-\infty}^{+\infty} dt\, \bar{g}(t)\, s(at+b) \right|$$

$$\leqslant \int_{-\infty}^{+\infty} dt\, |g(t)|\, [s(a\,|t|) + s(|b|)]$$

$$\leqslant O(1)\, \|sg\|_{L^1(\mathbb{R})}\, s(a) + O(1)\, \|g\|_{L^1(\mathbb{R})}\, s(b)$$

and the lemma is proved. ∎

The case o is treated by the next lemma.

Lemma 3.7. Let r be a nonnegative, even, submultiplicative function, and let s be a function that satisfies:

(i) $|s(t)| \leqslant cr(t)$ for all t.

(ii) $s(t) = o(r(t))$, $(t \to 0)$.

Then

$$\mathcal{W}_g s(b,a) = o(r(b) + r(a)) \qquad (b, a \to 0)$$

The wavelet should again satisfy $g \in L^1(\mathbb{R})$, $rg \in L^1(\mathbb{R})$.

Proof. For every $\varepsilon > 0$ we can find an $\eta > 0$ such that $|s(t)| < \varepsilon\, |r(t)|$ for $|t| < \eta$. Splitting the integral into two parts, we may write

$$|\mathcal{W}_g s(b,a)| \leqslant \int_{-\infty}^{+\infty} dt\, |\bar{g}(t)| \cdot |s(at+b)|$$

$$= \left\{ \int_{|at+b|<\eta} + \int_{|at+b|\geqslant\eta} \right\} dt\, |\bar{g}(t)| \cdot |s(at+b)| = X_1 + X_2$$

In the first term we have by hypothesis on η and submultiplicativity of r

$$X_1 \leqslant \varepsilon O(1) \int_{|at+b|<\eta} dt\, |\bar{g}(t)|\, [r(a)\, r(t) + r(b)]$$

$$\leqslant \varepsilon O(1)\, \|rg\|_{L^1(\mathbb{R})} r(a) + \varepsilon O(1)\, \|g\|_{L^1(\mathbb{R})}\, r(b)$$

$$\leqslant \varepsilon O(r(a) + r(b))$$

The second term may be estimated using the global estimation of s,

$$X_2 \leqslant o(1) \int_{|at+b| \geqslant \eta} dt\, |\bar{g}(t)|\, [r(a)\, r(t) + r(b)]$$

$$\leqslant O(r(b) + r(a)) \int_{|at+b| \geqslant \eta} dt\, |g(t)|\, [r(t) + 1]$$

Now $|g(t)|\, [r(t) + 1]$ is integrable, and therefore for a and b small enough the integral is smaller than ε, since it runs over a smaller and smaller neighborhood of infinity, and thus

$$X_2 < \varepsilon O(r(b) + r(a))$$

Since ε was arbitrary, the lemma is proved. ▮

This also shows the theorem. ▮

3.1. Inverse Theorems for Global Regularity

Here we shall prove some inverse theorems concerning the proof of global regularity through the wavelet coefficients. The general setting will be as follows. Suppose we have a position-scale representation of a distribution s,

$$s(t) = \int_0^\infty \frac{da}{a} \int_{-\infty}^{+\infty} db\, \mathscr{T}(b, a) \frac{1}{a} h\left(\frac{t-b}{a}\right)$$

with some polynomial-bounded scale-position coefficients

$$|\mathscr{T}(b, a)| \leqslant c(1 + |b|)^n\, (a + 1/a)^n$$

We then may split s into two parts,

$$s(t) = \int_0^\infty \frac{da}{a} \int_{-\infty}^{+\infty} db\, \mathscr{T}(b, a) \frac{1}{a} h\left(\frac{t-b}{a}\right)$$

$$= \left\{\int_0^1 \frac{da}{a} + \int_1^\infty \frac{da}{a}\right\} \int_{-\infty}^{+\infty} db\, \mathscr{T}(b, a) \frac{1}{a} h\left(\frac{t-b}{a}\right)$$

$$= s_{\text{small}}(t) + s_{\text{large}}(t)$$

Now the large-scale reconstruction s_{large} is a smooth, polynomial-bounded function. Therefore the local behavior of s is only determined by the small-scale behavior of $\mathscr{T}$. In the following theorem we show that uniform

Hölder regularity and similar ones may be completely characterized by the decay of the modulus of the scale-space coefficients as the scale gets small.

Theorem 3.8. Let $\mathcal{T}$ be some scale-space coefficient of s. Suppose that for large a, $\mathcal{T}$ is rapidly decreasing. Then in the limit $a \to 0$ we obtain the following classification:

$$|\mathcal{T}(b, a)| \leqslant ca^{\alpha} \Rightarrow s \in \Lambda^{\alpha}(\mathbb{R})$$

$$|\mathcal{T}(b, a)| = o(a^{\alpha}) \Rightarrow s \in \lambda^{\alpha}(\mathbb{R})$$

$$|\mathcal{T}(b, a)| \leqslant ca^{\alpha} \log^{\beta} a \Rightarrow s \in \Lambda_{\log}^{\alpha,\beta}(\mathbb{R})$$

$$|\mathcal{T}(b, a)| = o(a^{\alpha} \log^{\beta} a) \Rightarrow s \in \lambda_{\log}^{\alpha,\beta}(\mathbb{R})$$

The reconstruction wavelet should be compactly supported and $[\alpha]+1$ times[3] continuously differentiable.

Together with Theorem 3.3 this shows that the uniform regularity with nonintegral regularity exponent may be completely characterized through the small-scale behavior of the absolute value of the wavelet coefficients. The proof is again a corollary of the two more general lemmas that show that actually this kind of uniform local regularity analysis applies to submultiplicative regularities in general.

Theorem 3.9. Let r be a nonnegative, monotone, even, submultiplicative function that satisfies for some $n \in \mathbb{N}_0$:

(i) $\displaystyle\int_0^1 \frac{dt}{t^{1+n}} r(t) < \infty.$

(ii) $\displaystyle\int_1^{\infty} \frac{dt}{t^{2+n}} r(t) < \infty.$

Let $\mathcal{T}$ be some scale-space coefficients of some function s. Suppose $\mathcal{T}(\cdot, a) = 0$ for $a > 1$. Then if

$$|\mathcal{T}(b, a)| \leqslant cr(a)$$

it follows that s is n-times continuously differentiable and its derivative satisfies uniformly

$$|\partial^n s(t+u) - \partial^n s(t)| \leqslant O(r(u)/u^n)$$

[3] We denote by $[t]$ the biggest integer $\leqslant t$.

If the space-scale coefficients satisfy in addition

$$|\mathscr{T}(b, a)| = o(r(a))$$

we have uniformly in t

$$|\partial^n s(t+u) - \partial^n s(t)| = o(r(u)/u^n) \qquad (u \to 0)$$

In both cases the reconstruction wavelet is supposed to be compactly supported, having $n+1$ continuous derivatives.

Condition (i) ensures a certain decay at small scales. Condition (ii) ensures that this decay is not too fast. Indeed, because of the submultiplicativity of r we have $r(t) \geqslant c/r(1/t)$. Now by (ii) and monotony of r we have $t^{1+n} r(t) \to 0$ as $t \to \infty$. Therefore we have $t^{-n-1} r(t) \to \infty$ as $t \to 0$, or what is the same

$$t^{1+n} = o(r(t)) \qquad (t \to 0\ )$$

Thus in some sense conditions (i) and (ii) ensure that the local fluctuation is "bounded away" form the polynomial behavior, which would be $\sim t^n$ and $\sim t^{n+1}$.

Before we come to the proof, we recall the scheme of finite differences. Let Δ be the following operator:

$$\Delta\colon \quad s(t) \mapsto t^{-1}(s(t) - s(0))$$

Then we say the nth difference quotient of s exists at $t = 0$ iff

$$\Delta\Delta \cdots \Delta s(t)$$

converges to some finite number as $t \to 0$. We usually write Δ^n for the above expression. The nth difference quotient exists at 0 if there is polynomial of degree n such that

$$s(t) = P_n(t) + O(r(t))$$

with some $r(t) = o(t^n)$. In addition, this estimate is equivalent to having that the following estimate holds:

$$|\Delta^n(t) - \Delta^n(0)| \leqslant O(r(t)/t^n)$$

For later use we note the following communication relations:

$$\Delta^n T_b = T_b \Delta^n, \qquad \Delta^n D_a = a^{-n} D_a \Delta^n$$

In particular we have for $h_{b,a} = T_b D_a h$

$$\Delta^n h_{b,a} = a^{-n} (\Delta^n h)_{b,a}$$

Therefore if h is n-times continuously differentible and compactly supported, we have in particular

$$\|\Delta^n h_{b,a}\|_{L^1(\mathbb{R})} \leqslant c a^{-n} \tag{3.2}$$

with some c not depending on b or a.

Proof. By translation invariance it is enough to analyze s around 0. By an overall rescaling we may suppose that the support of h is contained in $[-1/2, +1/2]$. We only need to consider the case $t > 0$, since the case $t < 0$ is analogous. By hypothesis on s we may write pointwise $(0 < t \leqslant 1)$

$$\begin{aligned} s(t) &= \left\{ \int_0^t \frac{da}{a} + \int_t^1 \frac{da}{a} \right\} \int_{-\infty}^{+\infty} db \, \frac{1}{a} h\left(\frac{(-b)}{a} t \right) \mathscr{T}(b, a) \\ &= X_1(t) + X_2(t) \end{aligned}$$

We now prove the O part.

X_1. Using the decrease of the wavelet coefficients $\mathscr{T} = O(r(a))$ at small scales, we may write

$$\begin{aligned} |X_2| &\leqslant \int_0^t \frac{da}{a} \int_{-\infty}^{+\infty} db \left| \frac{1}{a} h\left(\frac{t-b}{a} \right) \right| |\mathscr{T}(b, a)| \\ &= O(1) \, \|h\|_{L^1(\mathbb{R})} \int_0^t \frac{da}{a} r(a) \\ &= O(r(t)) \end{aligned}$$

Because of condition (i) we certainly have $r(t) = o(t^n)$ as $t \to 0$ and thus $X_1(t)$ has a finite differential quotient of order n at 0. In the last equation we have used the fact that, because of the submultiplicativity of r, we have

$$\int_0^t \frac{da}{a} r(a) = \int_0^1 \frac{da}{a} r(at) \leqslant O(r(t)) \int_0^1 \frac{da}{a} r(a) = O(r(t)) \tag{3.3}$$

X_2. Since h is $n+1$ times continuously differentiable, we may write for $h_{b,a} = T_b D_a h$

$$\Delta^n h_{b,a}(t) - \Delta^n h_{b,a}(0) = t \Delta \Delta^n h_{b,a}(t)$$

In particular from (3.2) we have

$$\int_{-\infty}^{+\infty} db\,|\Delta^n h_{b,a}(t) - \Delta^n h_{b,a}(0)| \leqslant O(ta^{-n-1})$$

Thus we may write

$$\begin{aligned}|\Delta^n X_2(t) - \Delta^n X_2(0)| &\leqslant \int_t^1 \frac{da}{a}\int_{-\infty}^{+\infty} db\,|\Delta^n h_{b,a}(t) - \Delta^n h_{b,a}(0)|\cdot|\mathscr{T}(b,a)|\\ &\leqslant O(t)\int_t^1 \frac{da}{a}\,a^{-n}r(a)\\ &\leqslant O\left(\frac{r(t)}{t^n}\right)\end{aligned}$$

In the last equation we have again used the submultiplicativity of r via

$$\begin{aligned}t\int_t^1 \frac{da}{a^{2+n}}\,r(a) &= \int_1^{1/t} \frac{da}{a^{2+n}}\,r(a)\,t^{-n}\\ &\leqslant O\left(\frac{r(t)}{t^n}\right)\int_1^{1/t}\frac{da}{a^{2+n}}\,r(a) = O\left(\frac{r(t)}{t^n}\right) \qquad (3.4)\end{aligned}$$

since by hypothesis on r the integral is convergent.

All this taken together shows that $\Delta^n s$ is continuous and satisfies the stated estimates.

To prove the o part of the lemma, we remark that it is enough to replace estimations (3.3) and (3.4) by their o version. Suppose therefore that v is a nonnegative function that satisfies

$$v(t) = o(r(t)) \qquad (t \to 0)$$

Then for every $\varepsilon > 0$ we can find an η, $0 < \eta < 1$, such that $|v(t)| < \varepsilon r(t)$ whenever $0 \leqslant t \leqslant \eta$. Therefore for t small enough

$$\int_0^t \frac{da}{a}\,v(a) \leqslant \varepsilon\int_0^t \frac{da}{a}\,r(a) \leqslant \varepsilon r(t)$$

showing, because ε was arbitrarily chose, the o analog of (3.3).

For the analog of (3.4) we obtain, for fixed ε and t small enough with the same η as previously,

$$t\int_t^1 \frac{da}{a^{2+n}}\,v(a) = t\left\{\int_t^\eta + \int_\eta^1\right\}\frac{da}{a^{2+n}}\,v(a) = \sigma_1(t) + \sigma_2(t)$$

Now by submultiplicativity of r we have as before

$$\sigma_2(t) \leqslant O(t) \int_\eta^1 \frac{da}{a^{2+n}} v(a) \leqslant O\left(\frac{r(t)}{t^n}\right) \int_{\eta/t}^{1/t} \frac{da}{a^{2+n}} r(a)$$

By condition (ii) the integral tends to 0 as $t \to \infty$ and thus $\sigma_2(t) = o(r(t)/t^n)$.

By hypothesis (ii) on r we have for t small enough as in (3.4)

$$\sigma_1(t) \leqslant t \int_t^\eta \frac{da}{a^{2+n}} v(a) \leqslant \varepsilon t \int_t^1 \frac{da}{a^{2+n}} r(a) \leqslant \varepsilon O\left(\frac{r(t)}{t^n}\right)$$

Since ε was arbitrary, the theorem follows. ∎

3.2. The Class of Zygmund

Until now we had to exclude the case $\Lambda^\alpha(\mathbb{R})$ with $\alpha \in \mathbb{N}$. We will show now that the appropriate global regularity class of functions that can be analyzed by the wavelet transform in the case of integer exponents is the class of Zygmund.

Definition 3.10. A function s is in the class of Zygmund—written $\Lambda^*(\mathbb{R})$—if it is continuous and if it satisfies

$$|s(t_0 + t) + s(t_0 - t) - 2s(t_0)| \leqslant c\,|t| \tag{3.5}$$

uniformly in t_0. We may s is in $\lambda^*(\mathbb{R})$ if the same estimate holds with O replaced by o.

Functions in $\Lambda^*(\mathbb{R})$ do not have cusps, as can be seen from the identity

$$s(t_0 + t) + s(t_0 - t) - 2s(t_0) = (s(t_0 + t) - s(t_0)) - (s(t_0) - s(t_0 - t))$$

Therefore, e.g., the function $|t| \log|t|$ is not in $\Lambda^*(\mathbb{R})$. However, the function $t \log|t|$ is in the class of Zygmund. We now show how to characterize these regularity classes in wavelet space.

Theorem 3.11. Let s in the class of Zygmund $\Lambda^*(\mathbb{R})$. Then

$$\mathscr{W}_g s(b, a) = O(a) \qquad (a \to 0)$$

If $s \in \lambda^*(\mathbb{R})$, then the same estimate holds with O replaced by o. We suppose that g is in $S_0(\mathbb{R})$.

Remark. The assumption on g is much too strong and for technical convenience only. In the proof we will see how it can be relaxed.

Proof. First suppose that g is even,

$$g(t) = g(-t)$$

Then we have by symmetry and $\int_{-\infty}^{+\infty} dt\, g(t) =$ as usual

$$\mathscr{W}_g s(b, a) = \int_{-\infty}^{+\infty} dt \frac{1}{a} \bar{g}\left(\frac{t}{a}\right) s(t+b)$$

$$\times \frac{1}{2} \int_{-\infty}^{+\infty} dt \frac{1}{a} \bar{g}\left(\frac{t}{a}\right) \{a(t+b) + s(-t+b) - 2s(b)\}$$

Using the estimate (3.5) and the fact that s is bounded, we may estimate the parenthesis by $O(t)$ and the theorem follows. Suppose that h is not symmetric. The wavelet transform with respect to h is obtained from the one with respect to g by the action of the reproducing kernel. This function is localized enough and thus it does not change the decrease of the wavelet coefficients at small scale (see, e.g., Section 4 below).

The proof of the o part is analogous. ∎

But also the inverse theorem holds.

Theorem 3.12. Let s be a function with position-scale coefficients $\mathscr{T}$ supported by $a \leqslant 1$ that satisfy

$$\mathscr{T}(b, a) = O(a)$$

uniformly in b. Then $s \in \Lambda^*(\mathbb{R})$. If the same estimate holds with O replaced by o, then $s \in \lambda^*(\mathbb{R})$. The reconstruction wavelet should be two-times continuously differentiable and compactly supported.

Proof. By translation invariance it is enough to estimate

$$s(t) + s(-t) - 2s(0)$$

To each term corresponds an integral over some influence cone of h. The integrals over the region $0 < a \leqslant t$ may be estimated as previously. In the only new term X_3, say, we may wite, using the regularity of h,

$$h\left(\frac{t-b}{a}\right) + h\left(\frac{t-b}{a}\right) - 2h\left(-\frac{b}{a}\right) = t^2 \frac{1}{a^2} h''\left(\frac{\tilde{t}-b}{a}\right)$$

with some $\tilde{t} \in [0, t]$. We therefore end up with

$$X_3 = \int_t^1 \frac{da}{a} \int_{-\infty}^{+\infty} db$$

$$\times \frac{1}{a} \left\{ h\left(\frac{t-b}{a}\right) + h\left(\frac{t-b}{a}\right) - 2h\left(-\frac{b}{a}\right) \right\} \mathscr{T}(b, a)$$

$$= O(t^2) \, \|h''\|_{L^1(\mathbb{R})} \int_t^1 \frac{da}{a} a^{-1} = O(\eta)$$

The proof of the o part is similar. ▮

By what we have shown so far this implies via the characterization of the Hölder spaces through wavelet transforms the following for the regularity of the Zygmund class. We have

$$\Lambda^1(\mathbb{R}) \Rightarrow \Lambda^*(\mathbb{R}) \Rightarrow \Lambda^\alpha(\mathbb{R})$$

for $0 < \alpha < 1$. But, as follows easily from the proof of Theorem 3.9, we even have a little more precise information: a function in $\Lambda^*(\mathbb{R})$ satisfies

$$|s(t_0 + t) - s(t_0)| = O(t \log t) \qquad (t \to 0)$$

Indeed the logarithmic correction comes from the term X_2, as can be easily seen.

3.3. Inverse Theorems for Local Regularity

As we have seen, a local Hölder regularity implies a local decrease of the wavelet coefficients. On the other hand, as we just have proved, an overall decrease of the scale-space coefficients proves a global regularity of the reconstructed function. However, to prove local regularity through scale-space coefficients, we must suppose some stronger conditions.

Theorem 3.13. Let s be a scale-space representation $\mathscr{T}$ that satisfies at small scale $a < 1$ for some for some $\gamma > 0$:

(i) $\mathscr{T}(b,a) = O(a^\gamma)$ uniformly in b.

(ii) $\mathscr{T}(\tau_0 + b, a) = O(a^\alpha) + O(b^\alpha/\log b)$ $(b, a \to 0)$.

At large scale we suppose that $\mathscr{T}$ is rapidly decaying. Then s has at τ_0 a local regularity exponent α in the sense that

$$|s(\tau_0 + u) - P_n(u)| = O(u^\alpha) \qquad (u \to 0)$$

with some polynomial of order n, $\alpha - 1 < n < \alpha$.

Again this theorem holds for submultiplicative remainders in general.

Theorem 3.14. Let r be a nonnegative, submultiplicative function that satisfies for some $n \in \mathbb{N}_0$:

(i) $\int_0^1 \frac{dt}{t^{n+1}} r(t) < \infty$

(ii) $\int_1^\infty \frac{dt}{t^{n+2}} r(t) < \infty$

Let s have scale-space coefficients $\mathscr{T}$ supported by $a \leqslant 1$ that satisfy with some $\gamma > 0$:

(iii) $\mathscr{T}(b, a) = O(a^\gamma)$ uniformly in b.

(iv) $\mathscr{T}(\tau_0 + b, a) = O(r(a)) + O(r(b)/\log r(b))$ $(b, a \to 0)$.

Then there is a polynomial P_n of degree n such that

$$s(\tau_0 + t) = P_n(t) + O(r(t)) \qquad (t \to 0)$$

If we have in addition

(iv′) $\mathscr{T}(\tau_0 + b, a) = o(r(a)) + o(r(b)/\log r(b))$ $(b, a \to 0)$

then

$$s(\tau_0 + t) = P_n(t) + o(r(t)) \qquad (t \to 0)$$

The reconstruction wavelet should be $n+1$ times continuously differentiable and compactly supported.

Proof. We may suppose that $\gamma < 1$. Then with the help of the function $\eta(t) = r(t)^{2/\gamma}$ we may split the integral into several parts:

$$s(t) = \int_0^{\eta(t)} \frac{da}{a} \int_{-\infty}^{+\infty} db \frac{1}{a} h\left(\frac{t-b}{a}\right) \mathscr{T}(b, a) \qquad (X_1)$$

$$+ \int_{\eta(t)}^{t} \frac{da}{a} \int_{-\infty}^{+\infty} db \frac{1}{a} h\left(\frac{t-b}{a}\right) \mathscr{T}(b, a) \qquad (X_2)$$

$$+ \int_t^1 \frac{da}{a} \int_{-\infty}^{+\infty} db \frac{1}{a} h\left(\frac{t-b}{a}\right) \mathscr{T}(b, a) \qquad (X_3)$$

The contributions of these terms will be estimated independently.

X_1. Using the global Hölder regularity of s, we may estimate $\mathcal{T} = O(a^\gamma)$, which leads us to

$$|X_1| \leqslant \int_0^{\eta(t)} \frac{da}{a} \int_{-\infty}^{+\infty} db \left| \frac{1}{a} h\left(\frac{t-b}{a}\right)\right| \cdot |\mathcal{T}(b, a)|$$
$$= O(1) \, \|h\|_{L^1(\mathbb{R})} \int_0^{\eta(t)} \frac{da}{a} a^\gamma$$

By the choice of η we obtain $X_1 = O(r(t))$.

X_2. Since h is compactly supported, we have under the integral $a \leqslant O(t)$ and $b \leqslant O(t)$. Therefore we may estimate

$$|\mathcal{T}(b, a)| = O(r(a)) + O(r(t)/\log r(t))$$

By the same methods as for the term X_1 in the proof of Theorem 3.9 we can show that the contribution of the first term is of order $O(r(t))$, and we thus end up with

$$|X_2| \leqslant \int_{\eta(t)}^{t} \frac{da}{a} \int_{-\infty}^{+\infty} db \left| \frac{1}{a} h\left(\frac{t-b}{a}\right)\right| \cdot |\mathcal{T}(b, a)|$$
$$= O(1) \frac{r(t)}{|\log r(t)|} \int_{\eta(t)}^{t} \frac{da}{a} \int_{-\infty}^{+\infty} db \left| \frac{1}{a} h\left(\frac{t-b}{a}\right)\right| + O(r(t))$$
$$= \|h\|_{L^1(\mathbb{R})} O(1) \frac{r(t)}{|\log r(t)|} \log\left(\frac{t}{\eta(t)}\right) + O(r(t))$$
$$= O(r(t))$$

by the choice of η and since by hypothesis (ii) on r we have $\log(t) = O(\log(r(t))$.

X_3. This term may be treated in the same way as X_2 in the proof of Theorem 3.13, and the O part of the theorem is done.

The o part follows by the same considerations as before. ∎

Corollary 3.15. If $\mathcal{T}$ satisfies

(i) $\mathcal{T}(b, a) = O(a^\gamma)$ with some small $\gamma > 0$

(ii) $\mathcal{T}(\tau_0 + b, a) \leqslant O(b^\alpha + a^\alpha)$ $(b, a \to 0)$

then $|s(\tau_0 + t) - s(\tau_0)| = O(t^\alpha \log(t))$ $(t \to 0)$.

Proof. The logarithm is exactly the contribution of the second term in the previous demonstration. ∎

3.4. Pointwise Differentiability and Wavelet Analysis

We now want to study pointwise differentiability with the help of wavelet transforms. As we have seen in Section 3, the wavelet transform of a polynomial-bounded function s that is differentiable in τ_0 satisfies

$$\mathcal{T}(\tau_0+b, a)=o(|b|+a)$$

The analyzing wavelet should satisfy $g\in L^1(\mathbb{R})$, $tg\in L^1(\mathbb{R})$, and $\int g=\int tg=0$. To generalize the setting, we will once more consider fluctuations around polynomial approximations

$$s(\tau_0+t)=P_n(t)+o(t^n) \qquad (t\to 0)$$

or what is the same

$$\Delta_n(t)=o(1) \qquad (t\to 0)$$

where $\Delta_{k+1}(t)=t^{-1}[\Delta_k(t)-\Delta_k(0)]$ and $\Delta_0(t)=s(\tau_0+t)$. In this case we say that the nth differential quotient of s exists in τ_0. In wavelet space this regularity implies a decrease of

$$\mathcal{W}_g s(b, a)=o(a^n+|b|^n) \qquad (b, a\to 0)$$

whenever the wavelet is localized enough enough and its first n moments vanish.

Now we will prove an inverse theorem that relates the local decrease of the scale-space coefficients of the function to the differentiability of the function itself. We will see that it is not possible to prove the full inversion of the preceding, but a slightly stronger hypothesis on the wavelet side is needed to prove the differentiability of a function through its scale-space coefficients.

Theorem 3.16. If the scale-space coefficients $\mathcal{T}(b, a)$, $\mathcal{T}=0$ for $a>1$, of some function s satisfy

(i) $\mathcal{T}(b, a)=O(a^\gamma)$ uniformly in b

(ii) $\mathcal{T}(\tau_0+b, a)=O(r(a)+r(b))$

with an arbitrary $\gamma>0$ and with a nonnegative, monotone function $r=r(|a|)$ satisfying the Dini condition

$$\int_0^1 \frac{da}{a^{n+2}}\, r(a)<\infty$$

then the nth differential quotient of s exists in τ_0. Furthermore, the condition on r is optimal. The reconstruction wavelet h should be $n+1$ times continuously differentiable and compactly supported.

Proof. As usual we suppose $\tau_0=0$, $t>0$, and $\gamma<1$. As in the proof of Theorem 3.13, we only need to estimate the small-scale integral. Again with the help of the function $\eta(t)=r(t)^{2/\gamma}$, we may split the reconstruction integral into several parts,

$$s(t)=\int_0^{\eta(t)}\frac{da}{a}\int_{-\infty}^{+\infty}db\,\frac{1}{a}\,h\left(\frac{t-b}{a}\right)\mathscr{T}(b,a) \qquad (X_1)$$

$$+\int_{\eta(t)}^{t}\frac{da}{a}\int_{-\infty}^{+\infty}db\,\frac{1}{a}\,h\left(\frac{t-b}{a}\right)\mathscr{T}(b,a) \qquad (X_2)$$

$$+\int_{t}^{1}\frac{da}{a}\int_{-\infty}^{+\infty}db\,\frac{1}{a}\,h\left(\frac{t-b}{a}\right)\mathscr{T}(b,a) \qquad (X_3)$$

They will be estimated independently.

As in the proof of Theorem 3.14, we have $X_1(t)$, $X_2(t)=O(r(t))=o(t^n)$. Hence the nth differential quotient exists at $t=0$ and we only need to consider $\Delta^n X_3$. As for X_2 in the proof of Theorem 3.9, we have

$$|\Delta^n X_3(t)-\Delta^n X_3(0)|\leqslant O(t)\int_t^1\frac{da}{a^{2+n}}\,r(a)$$

and we have to show that this expression becomes arbitrarily small as $t\to 0$. Clearly $\lim_{a\mapsto 0} a^{-n-1}r(a)=0$ and therefore, given $\rho>0$, we can find $\delta>0$ such that $a^{-n-1}r(a)<\rho$ for $a<\delta$. Thus

$$t\int_t^1\frac{da}{a}\,a^{-n-1}r(a)\leqslant t\rho\int_t^{\delta}\frac{da}{a}+t\int_{\delta}^1\frac{da}{a}\,a^{-n}r(a)$$

The second term goes to 0 with t and the first term can be made arbitrarily small since ρ was arbitrary. Therefore the first part of the theorem is proved.

That the condition on r is actually the weakest possible can be seen as follows. We limit ourselves to $n=1$; the general case is similar. Consider a function g in the Schwarz class whose Fourier transform is supported by $[1,2]$ such that the functions $2^{j/2}g(2^j t)$, $j=1,2,\ldots$, are an orthonormal set. In addition we suppose that $g'(0)=1$. Then let

$$s(t)=\sum_{j=1}^{\infty}2^{-j}\varrho(2^{-j})\,g(2^j t)$$

with a continuous, positive, monotone, bounded function ϱ. The wavelet transform of s with respect to g satisfies

$$\mathscr{W}_g s(0, 2^{-j}) = 2^{-j} \varrho(2^{-j}), \qquad j = 1, 2, \ldots$$

Now s is differentiable for $t \neq 0$ and therefore $\Delta(t) = t^{-1}[s(t) - s(0)]$ can be written as

$$\Delta(t) = \sum_{j=1}^{\infty} \varrho(2^{-j})\, h'(2^j \tau)$$

with some $\tau = o(1)$ for $t \to 0$. Now ϱ satisfies a Dini condition if and only if

$$\sum_{j=1}^{\infty} \varrho(2^{-j}) < \infty$$

and therefore this same condition implies the differentiability of s at $t = 0$.

On the other hand, suppose that s is differentiable at $t = 0$. By an overall dilation we may suppose that $g'(t) \sim 1$ for $t \in [-1, 1]$. Then

$$\Delta(t) \sim \sum_{2^{-j} \geqslant \tau} \varrho(2^{-j}) + \sum_{2^{-j} < \tau} \varrho(2^{-j})\, g'(2^j \tau) = X_1 + X_2$$

Since ϱ is uniformly bounded and since we have

$$\sum_{2^{-j} < \tau} |g'(2^j \tau)| \leqslant \sup_{\xi \in [1/2, 1]} \sum_{j=1}^{\infty} |g'(2^j \xi)| < \infty$$

it follows that X_2 stays bounded as $t \to 0$. The same then must be be true for X_1 since $\Delta(t)$ stays bounded. This shows that ϱ satisfies a condition of Dini and so tha wavelet transform of s satisfies the condition of Theorem 3.16 if and only if s is differentiable in 0. ∎

We may even use the wavelet transform to compute the value of the derivative, if it exists. The following theorem is a generalization of a theorem due to Fatou.

Theorem 3.17. Let s be a periodic function or a measure. Suppose that at τ_0 the nth derivative $\partial_t s(\tau_0)$ exists.[4] Let $g \in L^1(\mathbb{R})$ and $t^n g \in L^1(\mathbb{R})$ be an analyzing wavelet that satisfies:

(i) $\displaystyle\int_{-\infty}^{+\infty} dt\, t^k g(t) = 0 \qquad \text{for} \quad k = 0, \ldots, n-1$

(ii) $\displaystyle\int_{-\infty}^{+\infty} dt\, t^n g(t) = {<} n! \Leftrightarrow (i\partial_\omega)^n \hat{g}(0) = 2\pi n!$

[4] By this we mean that the finite difference quotients have a limit.

Then

$$\lim_{a \to 0} a^{-n} \mathcal{W}_g s(\tau_0, a) = \partial_t^n s(\tau_0)$$

Proof. Writing $s(t) = P_{n-1}(t) + \partial_t^n \varepsilon(\tau_0)\, t^n/n! + o(t^n)$, we obtain

$$\mathcal{W}_g s(b, a) = \int_{-\infty}^{+\infty} dt\, \frac{1}{a}\, \bar{g}\left(\frac{t-b}{a}\right)\left(\frac{\partial_t^n s(\tau_0)\, t^n}{n!} + o(t^n)\right) = a^n \partial_t s(\tau_0) + o(a^n)$$

and the theorem follows. ▌

3.5. The Class W^α

As we have seen, global regularity can be characterized by a certain decrease of the modulus of the wavelet coefficients at small scale. Local regularity in the sense of fluctuations around a polynomial approximation could not be completely characterized: a little more is needed on the wavelet side. However, it is possible to define classes of local "wavelet regularity" that are characterized through the modulus of the wavelet coefficients. Let r be a monotone, nonnegative, submultiplicative function that satisfies some global estimate of the form

$$|r(t)| \leqslant O(1 + t^2)^{\gamma/2}$$

for some $\gamma > 0$.

Definition 3.18. We then say that a function (or distribution) s is of local WX^r regularity at τ_0 iff for some admissible wavelet $g \in S_0(\mathbb{R})$ we have

$$|\mathcal{W}_g s(\tau_0 + b, a)| \leqslant O(r(b) + r(a)) \qquad (b, a \to 0)$$

This is well-defined since we have the following results.

Theorem 3.19. The definition does not depend on the analyzing wavelet: if s is of regularity W^r with respect to some admissible $g \in S_0(\mathbb{R})$, it is of the same regularity with respect to any other $h \in S_0(\mathbb{R})$.

Proof. Indeed the wavelet transform of s with respect to $h \in S_0(\mathbb{R})$ is obtained through convolution over the half-plane with $\Pi = c_g \mathcal{W}_h g$, which is a highly localized function. Therefore we may exchange the integrals in the following expression and obtain

$$\int_0^{\infty} \frac{da'}{a'} \int_{-\infty}^{+\infty} db'\, \frac{1}{a'}\, \Pi\left(\frac{b-b'}{a'}, \frac{a}{a'}\right) r(b) = G_a * r(b)$$

with $G_a(t) = G(t/a)/a$ and

$$G(t) = \int_0^\infty \frac{da'}{a'} \Pi\left(\frac{t}{a'}, \frac{1}{a'}\right)$$

This function is again in $S_0(\mathbb{R})$ and thus by Lemma 3.6 this first expression is estimated by $O(r(b) + r(a))$.

In the same way we have

$$\int_0^\infty \frac{da'}{a'} \int_{-\infty}^{+\infty} db' \frac{1}{a'} \Pi\left(\frac{b-b'}{a'}, \frac{a}{a'}\right) r(a) = \int_0^\infty \frac{da'}{a'} H\left(\frac{a}{a'}\right) r(a')$$

where

$$H(a) = \int_{-\infty}^{+\infty} db\, \Pi(b, a)$$

This function is highly localized in the sense that

$$|H(a)| \leqslant O((a + 1/a)^{-\alpha})$$

for all $\alpha > 0$. We therefore have

$$\int_0^\infty \frac{da'}{a'} \left| H\left(\frac{a}{a'}\right)\right| r(a') = \int_0^\infty \frac{da'}{a'} H\left(\frac{1}{a'}\right) r(aa')$$
$$\leqslant O(r(a)) \int_0^\infty \frac{da'}{a'} H\left(\frac{1}{a'}\right) r(a')$$

which is again $O(r(a))$ because the integral is convergent. ▮

4. THE BROWNIAN MOTION

As a first application of the regularity analysis through wavelet transforms we analyze the regularity of a typical trajectory of a Brownian motion. In particular we want to redemonstrate the theorem of Lévi that states that with probability 1 it is of regularity $\Lambda_{\log}^{1/2, 1/2}$. Consider a modulated delta-comb

$$\gamma_\lambda(t) = \sum_{n=-\infty}^{+\infty} c_\lambda(n)\, \delta(t - \lambda n) \tag{4.1}$$

where the amplitudes c_λ are independent real-valued random variables.

Each of them is distributed according to a Gaussian law with mean value $\mu = 0$ and variance $\sigma_\lambda^2 > 0$,

$$\text{Prob}\{c_\lambda \leqslant t\} = \int_{-\infty}^{t} d\xi \, \frac{1}{(2\pi\sigma)^{1/2}} \, e^{-\xi^2/2\sigma_\lambda} \, d\xi$$

Consider a rapidly decaying continuous function. The action of the random measure (4.1) on s is itself a random variable

$$\gamma_\lambda(s) = \sum_{n=-\infty}^{+\infty} s(\lambda n)$$

Because the sum of Gaussian random variables with mean values μ_1 and μ_2 and variances σ_1^2 and σ_2^2 is again a Gaussian random variable with mean value $\mu = \mu_1 + \mu_2$ and variance $\sigma^2 = \sigma_1^2 + \sigma_2^2$, we have that $\mu_\lambda(s)$ is distributed with a Gaussian law with $\mu = 0$ and variance

$$\sigma^2 = \sigma_\lambda^2 \sum_{n=-\infty}^{+\infty} |s(\lambda n)|^2$$

Therefore if we choose $\sigma_\lambda = \sqrt{\lambda}$, the distribution of the random variable γ_λ will tend as $\lambda \to 0$ to a Gaussian distribution with

$$\mu = 0, \qquad \sigma^2 = \int_{-\infty}^{+\infty} dt \, |s(t)|^2$$

The limit random measure $\gamma_\lambda \to W$ is known as Wiener measure, or white noise. Consider now the primitive of the measures γ_λ,

$$F_\gamma(t) = \int_0^t \gamma_\lambda(du) = \sum_{0 < \lambda n \leqslant t} c_\lambda(n)$$

This function can be interpreted as a random walk with independent random increments at all points λn, $n \in \mathbb{Z}$. The limit random walk $\lambda \to 0$ exists and is known as Brownian motion,

$$B(t) = \int_0^t W(dt)$$

Now consider wavelet coefficients of the Wiener measure with respect to some orthonormal wavelet $g \in S_0(\mathbb{R})$,

$$\mathscr{W}_{j,k} = \mathscr{W}_g W(k2^{-j}, 2^{-j}) = \int_{-\infty}^{+\infty} W(dt) \, 2^j g(2^j t - k)$$

These numbers are again random variables. The correlation between two values of wavelet transform is given by the reproducing kernel. Therefore in the case of orthonormal wavelets these random variables are independently distributed. Each of them has a Gaussian distribution with

$$\mu_{j,k}=0, \qquad \sigma_{j,k}^2=2^j$$

The absolute value $|\mathscr{W}_{j,k}|$ has a χ distribution with density

$$\text{Prob}\{|\mathscr{W}_{j,k}|\leqslant t\}=\left(\frac{2}{\pi\sigma_{j,k}}\right)^{1/2}\int_0^t du\, e^{-u^2/2\sigma_{j,k}}=\left(\frac{2}{\pi}\right)^{1/2}\int_0^{2^{-j/2}t} du\, e^{-u^2/2}$$

We claim that from this it follows that given $c>-\log 2$ and given any interval $I\subset\mathbb{R}$, we can find with probability 1 an integer j_0 such that

$$|\mathscr{W}_{j,k}|\leqslant(c\,|j|)^{1/2}\,2^{j/2} \tag{4.2}$$

for all indices j, k such that $2^{-j}k\in I$ and $j\geqslant j_0$. Translated into the continuous wavelet transform language this implies that almost surely we have

$$\mathscr{W}_g W(b,a)=O(a^{-1/2}\log^{1/2}(a)) \qquad (a\to 0)$$

Thus for the Brownian motion—the primitive of W is essentially obtained through multiplication with a in wavelet space—we then have shown the following result.

Theorem 4.1. The Brownian motion is with probability one in the class $\Lambda_{\log}^{1/2,1/2}$, namely

$$\text{Prob}\left\{\limsup_{u\to+0}\frac{|B(t+u)-B(t)|}{[u\log(1/u)]^{1/2}}<\infty \text{ holds for all } t\in[0,1]\right\}=1$$

Proof. It remains to show that the above assertion (4.2) holds with probability one for j large enough and $k2^{-j}$ in some interval. By dilation covariance it is enough to consider the interval $[0, 1)$. Here we need the famous 0, 1 law of probability theory. It roughly states the following: let $x_1, x_2,\ldots$ be an inifinite family of independent random variables. Suppose Λ is an event that depends only on the infinite tail, namely on $x_n, x_{n+1},\ldots$ for n arbitrarily large. Then Λ occurs with probability 0 or with probability 1.

We start by noting that the random variables $|\mathscr{W}_{j,k}|$ with $2^{-j}k\in[0,1)$, $j\leqslant 0$, may be enumerated in such a way that the small-scale behavior is in the infinite tail of the sequence. Consider now the event that all numbers $|\mathscr{W}_{j,k}|$ satisfy the inequality above for al j large than some j_0. This event clearly depends only on the infinite tail of the ordered sequence of random variables. Its probability is therefore either 0 or 1. Let us compute the probability that $j_0 = 1$. This probability is a lower bound for the former probability. It is given by the infinite product

$$\text{Prob}\{|\mathscr{W}_{j,k}| \leqslant (cj)^{1/2}\, 2^{j/2}\} = \prod_{j=1}^{\infty} \left\{ \left(\frac{2}{\pi}\right)^{1/2} \int_0^{(cj)^{1/2}} du\, e^{-u^2/2} \right\}^{2^j}$$

Here we have use the fact that for a given j the 2^j random variables $|\mathscr{W}_{j,k}|$ with $2^{-j}k \in [0, 1)$ have the same probability law. From the asymptotic form of the integral

$$\int_t^{\infty} du\, e^{-u^2/2} = O\left(\frac{e^{-t^2}}{t}\right) \qquad (t \to \infty)$$

we see that the infinite product is convergent. Indeed convergence of the product is assured by the absolute convergence of the series

$$\sum_{j=1}^{\infty} \frac{1}{\sqrt{j}}\, 2^j e^{-cj} = \sum_{j=1}^{\infty} \frac{1}{\sqrt{j}}\, e^{-(c+\log 2)\,|j|}, \qquad c > -\log 2$$

Therefore for all $c > \log(1/2)$ we almost surely can find some j_0 such that the required estimation holds for all scales smaller than 2^{-j_0}. ∎

5. THE NONDIFFERENTIABLE FUNCTION OF WEIERSTRASS

In the year 1872 Weierstrass introduced his famous function

$$\sigma(t) = \sum_{n=1}^{\infty} \alpha^n \cos(\beta^n t), \qquad 0 < \alpha < 1$$

He could show that this function is continuous but nowhere differentiable whenever the product $\alpha \cdot \beta$ does exceed a certain value. Later Hardy showed that $\alpha \cdot \beta > 1$ is all that is needed. We now want to re-prove the result of Hardy with the help of the wavelet transform. It will become a two-line proof.

By computation in Fourier space we obtain

$$\mathscr{W}_g s(b, a) = \sum_{n=1}^{\infty} \alpha^n \hat{g}(\beta^n a)\, e^{i\beta^n b}$$

We may choose $g \in S_+(\mathbb{R})$ in such a way that $\operatorname{supp} \hat{g} \subset [1, \beta]$. Then the different frequencies decouple and we obtain

$$|\mathscr{W}_g s(b, a)| = \sum_{n=1}^{\infty} \alpha^n \, |\hat{g}(\beta^n a)|$$

Choosing $a_m = \beta^{-m}$, we see that this function does not decay with $o(a)$, thereby showing the nondifferentiability of σ. Instead we have

$$\mathscr{W}_g s(b, a) = O(a^{\log \beta/\log \alpha})$$

and this is the best possible estimation. Therefore $\sigma \in \Lambda^{\log \beta/\log \alpha}(\mathbb{R})$, but not in $\lambda^{\log \beta/\log \alpha}(\mathbb{R})$. In particular, σ is nowhere differentiable.

We may even choose $\alpha \cdot \beta = 1$. In this case we find that $\sigma \in \Lambda^*(\mathbb{R})$ but not in $\lambda^*(\mathbb{R})$, whence σ is still not differentiable.

Remark. Clearly the same argumentation applies to other lacunar Fourier series; that is, for sums of the kind

$$\sum_{k=1}^{\infty} \gamma_k \cos(\lambda_k t)$$

with $\lambda_k/\lambda_{k+1} > 1 + \varepsilon$ uniformly in k.

6. SELF-SIMILARITY AND THE RENORMALIZATION GROUP

We do not give a precise definition of what we call a fractal. Roughly speaking, a fractal is an object—for us a function or distribution—that has structure at all length scales. Therefore if we look at our object with a microscope more and more details will appear. Therefore these objects cannot be modeled by a smooth function. A smooth function looks at small scale essentially like a constant function. Very often fractals display a behavior known as self-similarity. This means that while looking at smaller and smaller scales we get the same features again and again. To be more preciose, consider the local functioon s_{loc} that describes the fluctuations of s around some point τ_0:

$$s_{\text{loc}}(t) = s(\tau_0 + t) - s(\tau_0)$$

Then local self-similarity would mean that for some rescaling factor λ we have in some sense

$$s_{\text{loc}}(\lambda t) \simeq c s_{\text{loc}}(t)$$

Therefore upon iterating we get with $\gamma = \lambda^n$

$$s_{\mathrm{loc}}(\gamma t) \simeq \gamma^{\alpha} s_{\mathrm{loc}}(t) \tag{6.1}$$

The only functions that satisfy this scaling invariance for all γ are the homogeneous functions. This motivates the following definitions already used in ref. 7.

Definition 6.1. A function s satisfies at τ_0 the exact scaling condition (ESC) if

$$s_{\mathrm{loc}}(t) = c_- \, |t|_-^{\alpha} + c_+ \, |t|_+^{\alpha} + o(t^{\alpha}) \qquad (t \to 0)$$

The exponent α is called the local scaling exponent at τ_0.

If instead Eq. (6.1) does not hold for all rescalings γ but only for $\gamma = \lambda^n$ we have another class of functions called the periodic scaling class.

Definition 6.2. A function s satisfies at τ_0 the periodic scaling condition (PSC) if

$$s_{\mathrm{loc}}(t) = c_-(t) \, |t|_-^{\alpha} + c_+(t) \, |t|_+^{\alpha} + o(t^{\alpha}) \qquad (t \to 0)$$

where $c_{\pm}$ satisfy the discrete scale invariance

$$c_{\pm}(\lambda t) = c_{\pm}(t)$$

with some constant $\lambda \geqslant 1$.

Remark 1. The number α is called the *local scaling exponent*, or *local fractal dimension*. It is actually a dimension, because it relates the t scale to the s scale, as, for instance, the "volume scale" is related to the "length scale" by a third power, which is the dimension of a volume.

Remark 2. We note that the exact scaling condition may be seen as a special case of the discrete scaling condition. In turn we may identify the periodic scaling condition with superpositions of functions satisfying the exact scaling condition, but with complex scaling exponent $\alpha + i\beta$.

Clearly the functions that satisfy PSC are locally self-similar in the sense described above. The greatest part of this section will be devoted to giving a sufficient criterion in wavelet space to show that the analyzed function actually satisfies ESC or PSC. The principal term can be obtained by direct computation. The remainder, however, is much more difficult to treat, as we shall see.

Sometimes the concepts of ESC and PSC are too strong and they shall be replaced by the following more general definition. We have defined the local function s_{loc} as the fluctuation of s around its value at a given point. However, to be more general, we consider the fluctuations of s around any local polynomial approximation of s; that is, if there is a polynomial P_n of order n, such that

$$s(\tau_0 + t) = P_n(t) + o(t^n) \qquad (t \to 0)$$

then we set for the highest such n

$$s_{\text{loc}}(t) = s(\tau_0 + t) - P(t)$$

Therefore to look at the fluctuations of a function around some local polynomial approximation amounts to considering the function modulo some polynomials. This can be done by taking s as distribution ion $S'_0(\mathbb{R})$.

To look at the fluctuations of s with a microscope can now be formalized as rescaling the t coordinate by $t \mapsto \lambda t$ with some $0 < \lambda < 1$ and rescaling the fluctuation scale of s in such a way that this process becomes eventually stabilized in a nontrivial way.

Definition 6.3. A function (or distribution) in S'_0 has a local renormalization (around 0) if there is a sequence λ_n, $0 < \lambda_n < 1$, and a sequence c_n such that the sequence

$$s_{n+1}(t) = c_n s_n(\lambda_n t), \qquad s_0(t) = s(t)$$

satisfies

$$\lim_{n \to \infty} s_n = s^* \neq 0$$

where the limit holds in $S'_0(\mathbb{R})$. The function s^* is called the local renormalization of s.

By translation we can define the renormalization around every point in the obvious way.

The local fractal dimension is again defined as the relation between the t scale and the fluctuation scale. That is, we set

$$\alpha = \alpha(t_0) = -\lim_{n \to \infty} \frac{\log c_n}{\log \lambda_n}$$

whenever this limit exists.

Usually the rescaling constants c_n are fixed by taking some function $\eta \in S_0(\mathbb{R})$—supposed to "measure" in some sense the fluctuation scale—and then by requiring that

$$\eta(s_n) = \langle \eta \mid s_n \rangle_{\mathbb{R}} = 1 \qquad \text{for all} \quad \eta \in \mathbb{N}_0 \tag{6.2}$$

This gives rise to a mapping of $S_0'(\mathbb{R})$ into itself,

$$\mathscr{R}_\lambda: \quad S_0'(\mathbb{R}) \to S_0'(\mathbb{R}), \qquad s \to cs(\lambda t)$$

where c is fixed by (6.2). This is called the renormalization map. It satisfies the semigroup property

$$\mathscr{R}_\lambda \circ \mathscr{R}_\gamma = \mathscr{R}_{\lambda \cdot \gamma}$$

To have a local renormalization with $\lambda_n = \lambda^n$ now means that s is in the attracting domain of some fixed point s^* of some renormalization map

$$\mathscr{R}_\lambda^n s \to s^* \quad (n \to \infty), \qquad \text{and} \qquad \mathscr{R}_\lambda s^* = s^*$$

Renormalization in Wavelet Space. We now want to compute the renormalization transformations in walet space. Suppose s has a local renormalization at $t_0 = 0$ with scale-fixing functional g. It is clear from the covariance of the wavelet transform under dilation that the renormalization procedure reads in wavelet space

$$\mathscr{W}_n(b, a) \mapsto c_n \mathscr{W}_n(\lambda_n b, \lambda_n a) \tag{6.3}$$

where the scale is fixed by

$$\mathscr{W}_n(\beta, \alpha) = 1 \qquad \text{for some} \quad (\beta, \alpha) \in \mathbb{H}$$

The convergence

$$\mathscr{W}_n(b, a) \to \mathscr{W}^*(b, a) \tag{6.4}$$

is pointwise. In addition, since, as we have seen before, the wavelet transform is continuous in the topology of $S_0(\mathbb{R})$, therefore $\mathscr{W}_g s^* = \mathscr{W}^*$.

Now consider the inverse problem. Suppose we have a sequence of numbers c_n and λ_n such that (6.4) converges. Does this imply that the analyzed function had a local renormalization? The answer is given by the following theorem.

Theorem 6.4. Suppose that the wavelet transform of a function $s \in S_0'(\mathbb{R})$ with respect to some admissible wavelet $g \in S_0(\mathbb{R})$ satisfies

$$\lim_{n \to \infty} \mathscr{W}_n(b, a) = \mathscr{W}^*(b, a) \not\equiv 0$$

pointwise for every $(b, a) \in \mathbb{H}$. Suppose further that for some m

$$|\mathscr{W}_n(b, a)| \leqslant (1 + |b|)^m (a + a^{-1})^m \qquad \text{uniformly in } n$$

Then s has a renormalization in $S_0'(\mathbb{R})$.

Proof. Let $r \in S_0(\mathbb{R})$. As we have seen in (2.5), we may write the action of $s_n(\cdot) = c^n s(\lambda^n \cdot)$ on any function $r \in S_+(\mathbb{R})$ as absolutely convergent integral over the half-plane,

$$s_n(r) = \int_0^\infty \frac{da}{a} \int_{-\infty}^{+\infty} db\, \mathscr{W}_n(b, a)\, \mathscr{W}_g r(b, a)$$

By hypothesis this function is uniformly majorized by an integrable function and we may invoke the dominated convergence theorem to write

$$s_n(r) \to \int_0^\infty \frac{da}{a} \int_{-\infty}^{+\infty} db\, \mathscr{W}^*(b, a)\, \mathscr{W}_g r(b, a)$$

This shows that s has a local renormalization in $S'_+(\mathbb{R})$. ∎

7. ASYMPTOTIC BEHAVIOR AT SMALL SCALES

In this section we give asymptotic expansions of the small-scale behavior of the wavelet coefficients. In particular we shall give a sufficient condition that s satisfies localy the exact scaling condition in Definition 6.1. Asymptotic expansions may be useful, because in general the wavelet transform at a given point depends on the analyzed function as well as on the analyzing wavelet, and in general it is difficult to say what is responsible for what. However, in the asymptotic limit of small length scales, as we shall see, we can clearly distinguish the influence of the analyzed function from that of the analyzing wavelet.

We recall the definition of an asymptotic series expansion of a function s. Let r_n, $n = 0, 1,...$, be a family of functions that satisfy

$$r_{n+1}(t) = o(r_n(t)) \qquad (t \to 0)$$

For instance, the family $r_n(t) = t^n$ will work. Then we say s has an asymptotic expansion of order N, in terms of r_n, if there are coefficients c_n such that we have

$$s(t_0 + t) = s(t_0) + \sum_{n=1}^{N} c_n r_n(t) + o(r_N(t)) \qquad (t \to 0)$$

and we write

$$s(t_0 + t) \simeq s(t_0) + \sum_{n=1}^{N} c_n r_n(t) \qquad (t \to 0)$$

We write

$$s(t_0+r) \simeq s(t_0) + \sum_{n=1}^{\infty} c_n r_n(t) \qquad (t \to 0)$$

If N may be chosen arbitrarily large. Note that the infinite series may diverge everywhere except for $t=0$. If a function s has an asymptotic expansion (finite or finite), the expansion coefficients c_n are uniquely determined by s.

As immediate application of the theorems of the preceding section we find that the following theorem holds:

Theorem 7.1. Let s be a polynomially bounded function. Suppose that s has the following infinite asymptotic expansion around t_0:

$$s(t_0+t) \simeq s(t_0) + \sum_{n=1}^{\infty} c_{+,n} |t|_+^{\alpha_n} + c_{-,n} |t|_-^{\alpha_n} + \sum_{n=1}^{\infty} \gamma_n t^n$$

with $\alpha_n \notin \mathbb{N}_0$, and $\Re\alpha_n$ monotonic growing in n and not constant as $n \to \infty$ and some arbitrary constants γ_n. Then the wavelet transform satisfies the following asymptotic expansion:

$$\mathscr{W}_g s(b, a) = \sum_{n=1}^{\infty} (c_{+,n} + c_{-,n} e^{i\pi\alpha_n})\, U(\alpha_n, b/a)\, a^{\alpha_n}$$

with

$$U(\alpha, u) = \int_0^{\infty} dt\, t^{\alpha} \bar{g}(t-u)$$

Suppose this expansion holds. If $s \in \Lambda^{\varepsilon}(\mathbb{R})$ for some $\varepsilon > 0$, it follows that s has the above local asymptotic expansion. The wavelet g is supposed to be progressive and highly time-frequency localized, $g \in S_+(\mathbb{R})$.

Thus the constants α_n, $c_{+,n}$, and $c_{-,n}$, and β_n may be determined from the asymptotic behavior of the wavelet transform. The polynomial behavior γ_n, however, is invisible in wavelet space.

Let us look at the principal part of the expansion ($\alpha = \alpha_n$)

$$\mathscr{W}_g s(t_0 + b, a) \simeq a^{\alpha} U_{\alpha}(b/a)$$

$$\begin{aligned} U_{\alpha}(u) &= (c_+ - c_- e^{i\pi\alpha}) \int_0^{\infty} dt\, t^{\alpha} \bar{g}(t-u) \\ &= -\frac{i\Gamma(\alpha+1)}{2\pi} (c_+ e^{-i\pi\alpha/2} - c_- e^{i\pi\alpha/2}) \int_0^{\infty} d\omega\, \omega^{-\alpha-1}\, \bar{\hat{g}}(\omega)\, e^{i\omega u} \end{aligned}$$

It corresponds to the local cusp

$$s(t_0+t) \simeq s(t_0) + P_n(t) + c_+ \, |t|_n^\alpha + c_- \, |t|_-^\alpha$$

where P is some polynomial.

First suppose that α is real-valued. Then if $\alpha \notin \mathbb{N}_0$ or if $\alpha \in \mathbb{N}_0$ but $c_- \neq (-1)^\alpha c_+$ the constants α, c_-, and c_+ can always be recovered from the asymptotic form of the wavelet coefficients. Upon a translation we may suppose that $t_0 = 0$. We then simply choose a line $b/a = c$ in the half-plane, passing through the location of the singularity such that for all α we have

$$\int_0^\infty dt \; t^\alpha \bar{g}(t-c) \neq 0$$

We say in this case that the zoom $b/a = c$ is allowed. For many wavelets every zoom will work; in the general case, however, it sqhould be checked. To fix the ideas, we suppose that $b/a = 0$ is an allowed zoom. We thus have at small scale

$$\mathcal{W}_g s(0, a) \simeq c e^{i\phi} a^\alpha \qquad (a \to 0), \qquad c \geqslant 0$$

If either c_+ or c_- is different from 0, then $c \neq 0$ and we obtain in a double logarithmic representation giving $\log(|\mathcal{W}_g s|)$ as a function of $\log a$ along this line asymptotically a straight line whose slope reveals the exponent α.

The small-scale behavior of the phase along this line is related to the phase of the complex number $c_+ e^{+i\alpha\pi/2} - c_- e^{-i\alpha\pi/2}$. Thus it determines the relative size of the constant c_+. Therefore if gives the qualitative aspects of the local singularity; that is, up to rescaling of the singularity. For simplicity, let us suppose that $\hat{g}$ is real-valued and positive. Then

$$\arg(c_+ e^{+i\alpha\pi/2} - c_- e^{-i\alpha\pi/2}) = \phi - \pi/2$$

The constant c determines the exponent via

$$|c_+ e^{+i\alpha\pi/2} - c_- e^{-i\alpha\pi/2}| = c$$

In case that the exponent is α is complex valued the modulus will show oscillations, that are related to the imaginary part of α.

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